

# The Dupire derivatives and Fréchet derivatives on continuous pathes \*

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**Abstract:** In this paper, we study the relation between Fréchet derivatives and Dupire derivatives, in which the latter are recently introduced by Dupire [4]. After introducing the definition of Fréchet derivatives for non-anticipative functionals, we prove that the Dupire derivatives and the extended Fréchet derivatives are coherent on continuous pathes.

**Keywords:** Dupire derivatives; Functional Itô's calculus; Backward stochastic differential equations; Path-dependent PDEs; Fréchet derivatives

## 1 Introduction

Recently Dupire [4] introduced the functional Itô's calculus, which was further developed in Cont and Fourni [1]-[3]. The key idea of Dupire [4] is to introduce the new "local" derivatives, i.e., horizontal derivative and vertical derivative for non-anticipative processes. Inspired by Dupire's work, Peng and Wang [10] obtained a nonlinear Feynman-Kac formula for classical solutions of path-dependent PDEs in terms of non-Markovian Backward stochastic differential equations (BSDEs for short). The viscosity solutions of path-dependent PDEs are also studied in [5] and [9] under this new framework. All these results show that Dupire derivative is an important tool to deal with functionals of continuous semimartingales.

The aim of this paper is to establish the relation between Dupire derivatives and Fréchet derivatives. Note that the Dupire derivative is a "local" one, in the sense that it is defined by perturbing the endpoint of a given current path. Compared with the Dupire derivative, the Fréchet derivative is defined by perturbing the whole path. Thus, it seems difficult to find the relationship between them.

To overcome the above difficulty, we introduce the definition of Fréchet derivatives for non-anticipative functionals. Inspired by Mohammed's work about stochastic functional differential

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\*This work was supported by National Natural Science Foundation of China (No. 11171187, No. 10871118 and No. 10921101); Supported by the Programme of Introducing Talents of Discipline to Universities of China (No.B12023); Supported by Program for New Century Excellent Talents in University of China.

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equations with bounded memory (see [7] and [8]), we study the weakly continuous linear and bilinear extensions of the Fréchet derivatives. By means of an auxiliary stochastic functional system, we show that the Dupire derivatives and the extended Fréchet derivatives are coherent on continuous pathes.

This paper is organized as follows. In section 2, we present some fundamental results of the Dupire derivatives and define the Fréchet derivatives for non-anticipative functionals. Furthermore, the unique extensions of the Fréchet derivatives are obtained. In section 3, under mild assumptions, we prove that the Dupire derivatives and the extended Fréchet derivatives are equal on continuous pathes.

## 2 Preliminaries

### 2.1 The Dupire derivatives

The following notations and tools are mainly from Dupire [4]. Let  $T > 0$  be fixed. For each  $t \in [0, T]$ , we denote by  $\Lambda_t$  the set of càdlàg  $\mathbb{R}^d$ -valued functions on  $[0, t]$ , and  $C$  is the set of continuous functions on  $[0, T]$ . For each  $\gamma(\cdot) \in \Lambda_T$  the value of  $\gamma(\cdot)$  at time  $s \in [0, T]$  is denoted by  $\gamma(s)$ . Thus  $\gamma(\cdot) = \gamma(s)_{0 \leq s \leq T}$  is a càdlàg process on  $[0, T]$  and its value at time  $s$  is  $\gamma(s)$ . The path of  $\gamma(\cdot)$  up to time  $t$  is denoted by  $\gamma_t$ , i.e.,  $\gamma_t = \gamma(s)_{0 \leq s \leq t} \in \Lambda_t$ . We denote  $\Lambda = \bigcup_{t \in [0, T]} \Lambda_t$ . For each  $\gamma_t \in \Lambda$  and  $x \in \mathbb{R}^d$  we denote by  $\gamma_t(s)$  the value of  $\gamma_t$  at  $s \in [0, t]$  and  $\gamma_t^x := (\gamma_t(s)_{0 \leq s < t}, \gamma_t(t) + x)$  which is also an element in  $\Lambda_t$ .

Let  $\langle \cdot, \cdot \rangle$  and  $|\cdot|$  denote the inner product and norm in  $\mathbb{R}^n$ . We now define a distance on  $\Lambda$ . For each  $0 \leq t, \bar{t} \leq T$  and  $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda$ , we denote

$$\begin{aligned} \|\gamma_t\| &:= \sup_{s \in [0, t]} |\gamma_t(s)|, \\ \|\gamma_t - \bar{\gamma}_{\bar{t}}\| &:= \sup_{s \in [0, t \vee \bar{t}]} |\gamma_t(s \wedge t) - \bar{\gamma}_{\bar{t}}(s \wedge \bar{t})|, \\ d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) &:= \sup_{0 \leq s \leq t \vee \bar{t}} |\gamma_t(s \wedge t) - \bar{\gamma}_{\bar{t}}(s \wedge \bar{t})| + |t - \bar{t}|. \end{aligned}$$

It is obvious that  $\Lambda_t$  is a Banach space with respect to  $\|\cdot\|$  and  $d_\infty$  is not a norm.

**Definition 2.1** A function  $u : \Lambda \mapsto \mathbb{R}$  is said to be  $\Lambda$ -continuous at  $\gamma_t \in \Lambda$ , if for any  $\varepsilon > 0$  there exists  $\delta > 0$  such that for each  $\bar{\gamma}_{\bar{t}} \in \Lambda$  with  $d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) < \delta$ , we have  $|u(\gamma_t) - u(\bar{\gamma}_{\bar{t}})| < \varepsilon$ .  $u$  is said to be  $\Lambda$ -continuous if it is  $\Lambda$ -continuous at each  $\gamma_t \in \Lambda$ .

**Definition 2.2** Let  $u : \Lambda \mapsto \mathbb{R}$  and  $\gamma_t \in \Lambda$  be given. If there exists  $p \in \mathbb{R}^d$ , such that

$$u(\gamma_t^x) = u(\gamma_t) + \langle p, x \rangle + o(|x|) \text{ as } x \rightarrow 0, \ x \in \mathbb{R}^d.$$

Then we say that  $u$  is (vertically) differentiable at  $\gamma_t$  and denote the gradient of  $\tilde{D}_x u(\gamma_t) = p$ .  $u$  is said to be vertically differentiable in  $\Lambda$  if  $\tilde{D}_x u(\gamma_t)$  exists for each  $\gamma_t \in \Lambda$ . We can similarly define the Hessian  $\tilde{D}_{xx}^2 u(\gamma_t)$ . It is an  $\mathbb{S}(d)$ -valued function defined on  $\Lambda$ , where  $\mathbb{S}(d)$  is the space of all  $d \times d$  symmetric matrices.

For each  $\gamma_t \in \Lambda$  we denote

$$\gamma_{t,s}(r) = \gamma_t(r)\mathbf{1}_{[0,t)}(r) + \gamma_t(t)\mathbf{1}_{[t,s]}(r), \quad r \in [0, s].$$

It is clear that  $\gamma_{t,s} \in \Lambda_s$ .

**Definition 2.3** For a given  $\gamma_t \in \Lambda$  if we have

$$u(\gamma_{t,s}) = u(\gamma_t) + a(s - t) + o(|s - t|) \text{ as } s \rightarrow t, \quad s \geq t,$$

then we say that  $u(\gamma_t)$  is (horizontally) differentiable in  $t$  at  $\gamma_t$  and denote  $\tilde{D}_t u(\gamma_t) = a$ .  $u$  is said to be horizontally differentiable in  $\Lambda$  if  $\tilde{D}_t u(\gamma_t)$  exists for each  $\gamma_t \in \Lambda$ .

**Definition 2.4** Define  $\mathbb{C}^{j,k}(\Lambda)$  as the set of function  $u := (u(\gamma_t))_{\gamma_t \in \Lambda}$  defined on  $\Lambda$  which are  $j$  times horizontally and  $k$  times vertically differentiable in  $\Lambda$  such that all these derivatives are  $\Lambda$ -continuous.

The following Itô formula was firstly obtained by Dupire [4] and then generalized by Cont and Fournié, [1], [2] and [3].

**Theorem 2.5** Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, P)$  be a probability space, if  $X$  is a continuous semimartingale and  $u$  is in  $\mathbb{C}^{1,2}(\Lambda)$ , then for any  $t \in [0, T]$ ,

$$\begin{aligned} u(X_t) - u(X_0) &= \int_0^t \tilde{D}_s u(X_s) ds + \int_0^t \tilde{D}_x u(X_s) dX(s) \\ &\quad + \frac{1}{2} \int_0^t \tilde{D}_{xx}^2 u(X_s) d\langle X \rangle(s), \quad P - a.s. \end{aligned}$$

## 2.2 The Fréchet Derivatives

Let  $C^*$  and  $C^\dagger$  be the space of bounded linear functionals  $\Phi : C \rightarrow R$  and bounded bilinear functionals  $\tilde{\Phi} : C \times C \rightarrow R$ , of the space  $C$ , respectively. They are equipped with the operator norms which will be, respectively, denoted by  $\|\cdot\|^*$  and  $\|\cdot\|^\dagger$ .

Fix  $t \in [0, T]$ . Let  $B_t = \{v\mathbf{1}_{\{t\}}, v \in R^n\}$ , where  $\mathbf{1}_{\{t\}} : [0, T] \rightarrow R$  is defined by

$$\mathbf{1}_{\{t\}}(s) := \begin{cases} 0, & \text{for } s \in [0, t), \\ 1, & \text{for } s = t, \\ 0, & \text{for } s \in (t, T]. \end{cases}$$

We define the direct sum

$$C \oplus B_t := \{\phi(\cdot) + v\mathbf{1}_{\{t\}} \mid \phi(\cdot) \in C, v \in R^n\}$$

and equip it with the norm  $\|\cdot\|$  defined by

$$\|\phi(\cdot) + v1_{\{t\}}\| = \sup_{s \in [0, T]} |\phi(s)| + |v|, \quad \phi(\cdot) \in C, v \in R^n.$$

For each  $\gamma(\cdot) \in C$  we denote

$$\gamma_t(s) = \begin{cases} \gamma(s), & s \leq t, \\ \gamma(t), & s > t. \end{cases}$$

It is clear that  $\gamma_t(\cdot) \in C$ .

**Definition 2.6** We call a functional  $\Psi : [0, T] \times C \mapsto R$  is non-anticipative, if for any  $t \in [0, T]$  and  $x(\cdot), y(\cdot) \in C$  satisfying the condition

$$y(\tau) = x(\tau) \quad \text{for } \tau \in [0, t],$$

there holds the equality

$$\Psi(t, x(\cdot)) = \Psi(t, y(\cdot)).$$

**Definition 2.7**  $\forall \gamma(\cdot) \in C$ , if we have

$$\Psi(s, \gamma_t(\cdot)) = \Psi(t, \gamma(\cdot)) + a(s - t) + o(|s - t|) \quad \text{as } s \rightarrow t, \quad s \geq t,$$

then we say that  $\Psi(t, \gamma(\cdot))$  is differentiable at  $t$  and denote  $D_t \Psi(t, \gamma(\cdot)) = a$ .

$\Psi$  is said to be differentiable in  $[0, T]$  if  $D_t \Psi(t, \gamma(\cdot))$  exists for each  $(t, \gamma(\cdot)) \in [0, T] \times C$ .

**Definition 2.8** For a given  $\gamma(\cdot) \in C$  and a non-anticipative  $\Psi$ , if we have

$$\Psi(t, \varphi(\cdot)) = \Psi(t, \gamma(\cdot)) + D_x \Psi(t, \gamma(\cdot))((\varphi(\cdot) - \gamma(\cdot))) + o(\|(\varphi(\cdot) - \gamma(\cdot))1_{[0, t]}\|),$$

for each  $\varphi(\cdot) \in C$ , then we say that  $\Psi(t, \gamma(\cdot))$  is Fréchet differentiable at  $\gamma(\cdot)$ .

$\Psi$  is said to be differentiable in  $C$  if  $D_x \Psi(t, \gamma(\cdot))$  exists for each  $(t, \gamma(\cdot)) \in [0, T] \times C$ .

**Remark 2.9** For a non-anticipative  $\Psi$ , if  $\Psi(t, \gamma(\cdot))$  is Fréchet differentiable at  $\gamma(\cdot)$ , then it is obvious

$$D_x \Psi(t, \gamma(\cdot))(\eta(\cdot)) = D_x \Psi(t, \gamma(\cdot))(\eta(\cdot)1_{[0, t]}), \quad \forall \eta(\cdot) \in C.$$

**Definition 2.10** Define  $C^{j, k}([0, T] \times C)$  as the set of non-anticipative functions  $\Psi$  defined on  $[0, T] \times C$  which are  $j$  times differentiable in time and  $k$  times Fréchet differentiable in  $C$  such that all these derivatives are continuous.

Using similar techniques as in Mohammed [7] and [8], we have the following lemma.

**Lemma 2.11** Suppose a non-anticipative  $\Phi : [0, T] \times C \rightarrow R$  is second order continuous differentiable. Then  $\forall \phi(\cdot) \in C$ , the Fréchet derivatives  $D_x \Phi(t, \phi(\cdot))$  and  $D_{xx}^2 \Phi(t, \phi(\cdot))$  have unique weakly continuous linear and bilinear extensions

$$\overline{D_x \Phi(t, \phi(\cdot))} \in (C \oplus B_t)^*, \quad \overline{D_{xx}^2 \Phi(t, \phi(\cdot))} \in (C \oplus B_t)^\dagger.$$

**Proof.** It is sufficient to consider the one-dimensional case, i.e.,  $n = 1$ .

For a fixed  $t \in [0, T)$  and  $\phi(\cdot) \in C$ , we will show that there is a unique weakly continuous extension  $\overline{D_x \Phi(t, \phi(\cdot))} \in (C \oplus B_t)^*$  of the first Fréchet derivatives  $D_x \Phi(t, \phi(\cdot))$ . In other words, if  $\{\xi^k\}$  is a bounded sequence in  $C$  such that  $\xi^k(s) \rightarrow \xi(s)$  as  $k \rightarrow \infty$  for all  $s \in [0, T]$  where  $\xi \in C \oplus B_t$ , then  $D_x \Phi(t, \xi^k(\cdot)) \rightarrow \overline{D_x \Phi(t, \xi(\cdot))}$  as  $k \rightarrow \infty$ . Note that  $\Phi$  is non-anticipative. Then for all  $\eta \in C$ ,

$$D_x \Phi(t, \phi(\cdot))(\eta(\cdot)) = D_x \Phi(t, \phi_t(\cdot))(\eta(\cdot)1_{[0, t]}).$$

By the Riesz representation theorem, there is a unique finite Borel measure  $\mu$  on  $[0, T]$  such that

$$D_x \Phi(t, \phi(\cdot))(\eta(\cdot)) = \int_0^t \eta(s) d\mu(s). \quad (2.1)$$

Define  $\overline{D_x \Phi(t, \phi(\cdot))} \in (C \oplus B_t)^*$  by

$$\overline{D_x \Phi(t, \phi(\cdot)) + v1_{\{t\}}} = D_x \Phi(t, \phi(\cdot)) + v\mu(t), \quad \eta \in C, \quad v \in R.$$

We know that  $\overline{D_x \Phi(t, \phi(\cdot))}$  is weakly continuous by Lebesgue's dominated convergence theorem. The weak extension  $\overline{D_x \Phi(t, \phi(\cdot))}$  is unique because for any  $v \in R$ , the function  $v1_{\{t\}}$  can be approximated weakly by a sequence of continuous functions  $\{\xi_0^k\}$ , where

$$\xi_0^k(s) := \begin{cases} (ks + 1)v, & -\frac{1}{k} + t \leq s \leq t \\ 0, & 0 \leq s < -\frac{1}{k} + t. \end{cases}$$

Similarly, we can construct a unique weakly continuous bilinear extension  $\overline{D_{xx}^2 \Phi(t, \phi(\cdot))} \in (C \oplus B_t)^\dagger$  for any continuous bilinear form  $D_{xx}^2 \Phi(t, \phi(\cdot))$ . ■

### 3 The relation between Dupire derivatives and Fréchet derivatives

In order to establish the relation between Dupire derivatives and Fréchet derivatives, we need the following auxiliary stochastic functional differential equation: for given  $t \in [0, T)$  and  $\gamma(\cdot) \in \Lambda_T$ ,

$$\begin{aligned} dX^{\gamma_t}(s) &= b(s, X^{\gamma_t}(\cdot))ds + \sigma(s, X^{\gamma_t}(\cdot))dW(s), \quad s \in [t, T], \\ X^{\gamma_t}(r) &= \gamma_t(r), \quad r \in [0, t], \end{aligned} \quad (3.1)$$

where  $\{W(s), s \in [0, T]\}$  is the  $d$ -dimensional standard Brownian motion; the process  $\{X^{\gamma_t}(s), 0 \leq s \leq T\}$  takes values in  $\mathbb{R}^n$ ;  $b : [0, T] \times C \rightarrow \mathbb{R}^n$  and  $\sigma : [0, T] \times C \rightarrow \mathbb{R}^n \times \mathbb{R}^d$  are non-anticipative functionals.

**Definition 3.1** A process  $\{X^{\gamma_t}(s), s \in [t, T]\}$  is said to be a strong solution of the equation (3.1) on the interval  $[t, T]$  and through the initial datum  $\gamma_t \in \Lambda$  if it satisfies the following conditions:

- (1)  $X_t^{\gamma_t} = \gamma_t$ ;
- (2)  $X^{\gamma_t}(s)$  is  $\mathcal{F}(s)$ -measurable for each  $s \in [t, T]$ ;
- (3) The process  $\{X^{\gamma_t}(s), s \in [t, T]\}$  is continuous and it satisfies the following stochastic integral equation  $P - a.s.$

$$X^{\gamma_t}(s) = \gamma_t(t) + \int_t^s b(r, X^{\gamma_t}(\cdot))dr + \int_t^s \sigma(r, X^{\gamma_t}(\cdot))dW(r).$$

We assume  $b, \sigma$  satisfy the following Lipschitz and bounded conditions.

**Assumption 3.2**  $b(\cdot, x(\cdot)), \sigma(\cdot, x(\cdot))$  are progressively measurable processes for each  $x(\cdot) \in C$ , and there exists a constant  $c > 0$  such that

$$|b(s, x^1(\cdot)) - b(s, x^2(\cdot))| + |\sigma(s, x^1(\cdot)) - \sigma(s, x^2(\cdot))| \leq c \|x_s^1(\cdot) - x_s^2(\cdot)\|,$$

$$\forall (s, x^1(\cdot)), (s, x^2(\cdot)) \in [0, T] \times C.$$

**Assumption 3.3** There exists a constant  $K > 0$  such that

$$|b(s, \Phi(\cdot))| + |\sigma(s, \Phi(\cdot))| \leq K, \quad \forall (s, \Phi(\cdot)) \in [0, T] \times C.$$

Then we have the following theorem (see [6]):

**Theorem 3.4** Under assumptions (3.2) and (3.3), the equation (3.1) has a unique strong solution.

By similar analysis as in Mohammed [7] and [8], we have the following result.

**Theorem 3.5** Let Assumptions (3.2) and (3.3) hold true.  $X^{\gamma_t}(\cdot)$  is the solution of (3.1). Suppose a non-anticipative  $\Phi$  belongs to  $C^{1,2}([0, T] \times C)$ . Then for given  $\gamma \in C$ ,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{E[\Phi(t+\varepsilon, X^{\gamma_t}(\cdot)) - \Phi(t, \gamma(\cdot))]}{\varepsilon} = & D_t \Phi(t, \gamma(\cdot)) + \overline{D_x \Phi(t, \gamma(\cdot))}(b(t, \gamma(\cdot))1_{\{t\}}) \\ & + \frac{1}{2} \sum_{j=1}^n \overline{D_{xx}^2 \Phi(t, \gamma(\cdot))}(\sigma(t, \gamma(\cdot))e_j 1_{\{t\}}, \sigma(t, \gamma(\cdot))e_j 1_{\{t\}}), \end{aligned} \quad (3.2)$$

**Proof.** Step 1.

Fix  $\gamma(\cdot) \in C$ . Since  $\Phi \in C^{1,2}([0, T] \times C)$ , by Taylor's theorem, for  $\varepsilon > 0$ ,

$$\begin{aligned} \Phi(t + \varepsilon, X^{\gamma_t}(\cdot)) - \Phi(t, \gamma(\cdot)) &= \Phi(t + \varepsilon, \gamma_t(\cdot)) - \Phi(t, \gamma(\cdot)) + \Phi(t + \varepsilon, X^{\gamma_t}(\cdot)) - \Phi(t + \varepsilon, \gamma_t(\cdot)) \\ &= D_t \Phi(t, \gamma(\cdot)) \cdot \varepsilon + D_x \Phi(t + \varepsilon, \gamma_t(\cdot))(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0, t+\varepsilon]} \\ &\quad + R(\varepsilon) + o(\varepsilon), \quad a.s. \end{aligned}$$

where

$$R(\varepsilon) := \int_0^1 (1-u) D_{xx}^2 \Phi(t+\varepsilon, \gamma_t(\cdot) + u \cdot (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))) \\ ((X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}, (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}) du.$$

Taking expectation and dividing by  $\varepsilon$ , we have

$$\frac{E[\Phi(t+\varepsilon, X^{\gamma_t}(\cdot)) - \Phi(t, \gamma(\cdot))]}{\varepsilon} = D_t \Phi(t, \gamma(\cdot)) + D_x \Phi(t+\varepsilon, \gamma_t(\cdot)) \cdot E[\frac{1}{\varepsilon}(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}] \\ + \frac{1}{\varepsilon} ER(t) + o(1). \quad (3.3)$$

Note that

$$\lim_{\varepsilon \rightarrow 0^+} [E\{\frac{1}{\varepsilon}(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}\}](s) = \begin{cases} \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} E[b(u, X^{\gamma_t}(\cdot))] du, & s = t \\ 0, & 0 \leq s < t \end{cases} \\ = b(t, \gamma(\cdot))1_{\{t\}}, \quad 0 \leq s \leq t.$$

Since  $b$  is bounded,  $\|E\{\frac{1}{\varepsilon}(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}\}\|_C$  is bounded at  $t$  and  $\gamma_t(\cdot) \in C$ . Hence

$$\lim_{\varepsilon \rightarrow 0^+} [E\{\frac{1}{\varepsilon}(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}\}] = b(t, \gamma(\cdot))1_{\{t\}}.$$

Therefore, by Lemma (2.11) and the continuity of  $D_x \Phi$  at  $\gamma(\cdot)$ , we obtain

$$\lim_{\varepsilon \rightarrow 0^+} D_x \Phi(t+\varepsilon, \gamma_t(\cdot)) [E\{\frac{1}{\varepsilon}(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}\}] \\ = \lim_{\varepsilon \rightarrow 0^+} D_x \Phi(t, \gamma(\cdot)) [E\{\frac{1}{\varepsilon}(X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}\}] \\ = \overline{D_x \Phi(t, \gamma(\cdot))} (b(t, \gamma(\cdot))1_{\{t\}}).$$

Step 2.

Finally we calculus the limit of the third term in the right-hand side of (3.3) as  $\varepsilon \rightarrow 0^+$ . By the martingale property of the Itô integral and the Lipschitz continuity of  $D_{xx}^2 \Phi$ , we have the following estimates:

$$| \frac{1}{\varepsilon} E D_{xx}^2 \Phi(t+\varepsilon, \gamma_t(\cdot) + u \cdot (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))) ((X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}, (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}) \\ - \frac{1}{\varepsilon} E D_{xx}^2 \Phi(t, \gamma(\cdot)) ((X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}, (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]}) | \\ \leq (E \| D_{xx}^2 \Phi(t+\varepsilon, \gamma_t(\cdot) + u \cdot (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))) - D_{xx}^2 \Phi(t, \gamma(\cdot)) \|^2)^{\frac{1}{2}} [\frac{1}{\varepsilon^2} E \| (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0,t+\varepsilon]} \|^4]^{\frac{1}{2}} \\ \leq K(\varepsilon^2 + 1)^{\frac{1}{2}} (E \| D_{xx}^2 \Phi(t+\varepsilon, \gamma_t(\cdot) + u \cdot (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))) - D_{xx}^2 \Phi(t, \gamma(\cdot)) \|^2)^{\frac{1}{2}},$$

where  $t \in R^+$ ,  $\gamma(\cdot) \in C$  and  $K$  is a positive constant independent of  $u$ . The last line tends to 0, uniformly for  $u \in [0, 1]$ , as  $\varepsilon \rightarrow 0^+$ . Because  $\Phi \in C^{1,2}([0, T] \times C)$  and is bounded on  $C$ , we

have the following weak limit:

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} ER(\varepsilon) &= \int_0^1 (1-u) \lim_{\varepsilon \rightarrow 0^+} \frac{1}{\varepsilon} ED_{xx}^2 \Phi(t, \gamma(\cdot)) ((X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0, t+\varepsilon]}, (X^{\gamma_t}(\cdot) - \gamma_t(\cdot))1_{[0, t+\varepsilon]}) \\ &= \frac{1}{2} \sum_{j=1}^n \overline{D_{xx}^2 \Phi(t, \gamma(\cdot))} (\sigma(t, \gamma(\cdot))e_j 1_{\{t\}}, \sigma(t, \gamma(\cdot))e_j 1_{\{t\}}).\end{aligned}$$

■

Note that  $b$  and  $\sigma$  are non-anticipative functionals. We can rewrite  $b(t, \gamma(\cdot)) = \tilde{b}(\gamma_t)$  and  $\sigma(t, \gamma(\cdot)) = \tilde{\sigma}(\gamma_t)$ ,  $\forall \gamma(\cdot) \in C$ .

**Corollary 3.6** *Let Assumptions (3.2) and (3.3) hold true.  $X^{\gamma_t}(\cdot)$  is the solution of (3.1).  $\Phi$  in  $\mathbb{C}^{1,2}(\Lambda)$  is non-anticipative. Then for any  $t \in [0, T]$ ,*

$$\lim_{\varepsilon \rightarrow 0^+} \frac{E[\Phi(X_{t+\varepsilon}^{\gamma_t})] - \Phi(\gamma_t)}{\varepsilon} = \tilde{D}_t \Phi(\gamma_t) + \langle \tilde{D}_x \Phi(\gamma_t), \tilde{b}(\gamma_t) \rangle + \frac{1}{2} \langle \tilde{D}_{xx} \Phi(\gamma_t) \tilde{\sigma}(\gamma_t), \tilde{\sigma}(\gamma_t) \rangle. \quad (3.4)$$

It is easy to prove this corollary by Theorem 2.5.

Now we build the relation between Fréchet derivatives and Duprie derivatives.

**Theorem 3.7** *Suppose (i)  $\Phi \in \mathbb{C}^{1,2}(\Lambda)$ . (ii) When the domain of  $\Phi$  is limited to  $[0, T] \times C$ , it is non-anticipative and belongs to  $C^{1,2}([0, T] \times C)$ . Then, for any given  $\gamma(\cdot) \in C$ , we have the following equalities:*

$$\begin{aligned}\tilde{D}_t \tilde{\Phi}(\gamma_t) &= D_t \tilde{\Phi}(\gamma_t), \\ \tilde{\mu}(t) &= \tilde{D}_x \tilde{\Phi}(\gamma_t), \\ \tilde{\lambda}(t) &= \tilde{D}_{xx}^2 \tilde{\Phi}(\gamma_t),\end{aligned}$$

where  $\tilde{\Phi}(\gamma_t) = \Phi(t, \gamma(\cdot))$ ,  $\tilde{\mu}$  and  $\tilde{\lambda}$  are the corresponding Borel measures of  $\overline{D_x \tilde{\Phi}(\gamma_t)}$  and  $\overline{D_{xx}^2 \tilde{\Phi}(\gamma_t)}$ .

**Proof.** For given  $\gamma(\cdot) \in C$ , we rewrite  $\tilde{\Phi}(X_s^{\gamma_t}) = \Phi(s, X^{\gamma_t}(\cdot))$ ,  $\tilde{b}(\gamma_t) = b(t, \gamma(\cdot))$  and  $\tilde{\sigma}(\gamma_t) = \sigma(t, \gamma(\cdot))$ . By Theorem (3.5), we have

$$\begin{aligned}\lim_{\varepsilon \rightarrow 0^+} \frac{E[\Phi(t+\varepsilon, X^{\gamma_t}(\cdot))] - \Phi(t, \gamma(\cdot))}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \frac{E[\tilde{\Phi}(X_{t+\varepsilon}^{\gamma_t})] - \tilde{\Phi}(\gamma_t)}{\varepsilon} \\ &= \Phi(t, \gamma(\cdot)) + \overline{D_x \Phi(t, \gamma(\cdot))} (b(t, \gamma(\cdot))1_{\{t\}}) \\ &\quad + \frac{1}{2} \sum_{j=1}^n \overline{D_{xx}^2 \Phi(t, \gamma(\cdot))} (\sigma(t, \gamma(\cdot))e_j 1_{\{t\}}, \sigma(t, \gamma(\cdot))e_j 1_{\{t\}}) \\ &= D_t \tilde{\Phi}(\gamma_t) + \overline{D_x \tilde{\Phi}(\gamma_t)} (\tilde{b}(\gamma_t)1_{\{t\}}) \\ &\quad + \frac{1}{2} \sum_{j=1}^n \overline{D_{xx}^2 \tilde{\Phi}(\gamma_t)} (\tilde{\sigma}(\gamma_t)e_j 1_{\{t\}}, \tilde{\sigma}(\gamma_t)e_j 1_{\{t\}}).\end{aligned} \quad (3.5)$$



Similar as the proof of lemma (2.11), we know there is a unique finite Borel measure  $\tilde{\mu}$  on  $[0, T]$  such that

$$D_x \tilde{\Phi}(\gamma_t)(\eta(s)) = \int_0^t \eta(s) d\tilde{\mu}(s). \quad (3.6)$$

Then we have

$$\overline{D_x \tilde{\Phi}(\gamma_t)}(\tilde{b}(\gamma_t)1_{\{t\}}) = \langle \tilde{\mu}(t), \tilde{b}(\gamma_t) \rangle,$$

There is also a unique finite Borel measure  $\tilde{\lambda}$  on  $[0, T]$  such that

$$\frac{1}{2} \langle \tilde{\lambda}(t) \tilde{\sigma}(\gamma_t), \tilde{\sigma}(\gamma_t) \rangle = \frac{1}{2} \sum_{j=1}^n \overline{D_{xx}^2 \tilde{\Phi}(\gamma_t)}(\tilde{\sigma}(\gamma_t) e_j 1_{\{t\}}, \tilde{\sigma}(\gamma_t) e_j 1_{\{t\}}).$$

It yields that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{E[\Phi(t+\varepsilon, X^{\gamma_t}(\cdot)) - \Phi(t, \gamma(\cdot))]}{\varepsilon} &= \lim_{\varepsilon \rightarrow 0^+} \frac{E[\tilde{\Phi}(X_{t+\varepsilon}^{\gamma_t}) - \tilde{\Phi}(\gamma_t)]}{\varepsilon} \\ &= D_t \tilde{\Phi}(\gamma_t) + \langle \tilde{\mu}(t), \tilde{b}(\gamma_t) \rangle + \frac{1}{2} \langle \tilde{\lambda}(t) \tilde{\sigma}(\gamma_t), \tilde{\sigma}(\gamma_t) \rangle. \end{aligned} \quad (3.7)$$

By Corollary (3.6), we have

$$\lim_{\varepsilon \rightarrow 0^+} \frac{E[\Phi(X_{t+\varepsilon}^{\gamma_t}) - \Phi(\gamma_t)]}{\varepsilon} = \tilde{D}_t \Phi(\gamma_t) + \langle \tilde{D}_x \Phi(\gamma_t), \tilde{b}(\gamma_t) \rangle + \frac{1}{2} \langle \tilde{D}_{xx} \Phi(\gamma_t) \tilde{\sigma}(\gamma_t), \tilde{\sigma}(\gamma_t) \rangle. \quad (3.8)$$

Notice that  $b$  and  $\sigma$  can take any values which satisfy Assumptions (3.2) and (3.3). Comparing (3.7) and (3.8), we obtain the results. ■

**Acknowledgement.** *The authors would like to thank Shige Peng for pointing out that based on our results, Dupire derivative is a concept weak than Fréchet derivative.*

## References

- [1] R. Cont and D. A. Fourni, Functional Itô calculus and stochastic integral representation of martingales, to appear in Annals of Probability, 2012.
- [2] R. Cont and D. A. Fourni, A functional extension of the Itô formula, C. R. Math. Acad. Sci. Paris 348 (2010), no. 1-2, 57-61.
- [3] R. Cont and D. A. Fourni, Change of variable formulas for non-anticipative functionals on path space, J. Funct. Anal. 259 (2010), no. 4, 1043-1072.
- [4] B. Dupire, Functional Itô calculus, Portfolio Research Paper, Bloomberg, 2009.
- [5] I. Ekren, C. Keller, N. Touzi and J. Zhang, On Viscosity Solutions of Path Dependent PDEs, to appear in Annals of Probability, 2012.
- [6] R. S. Lipster and A. N. Shiryaev, *Statistics of Random Processes I*, Springer, 1978.

- [7] S-E.A, Mohammed, Stochastic Functional Differential Equations. *Research Notes in Mathematics No.99* (Boston, London, Melbourne: Pitman Publishing), 1984.
- [8] S-E.A, Mohammed, Stochastic Differential Equations with memory—theory, examples and applications, Stochastic Analysis and Related Topics 6. The Geido Workshop, Progress in probability, Birkhauser, 1996.
- [9] S. Peng, Note on Viscosity Solution of Path-Dependent PDE and G-Martingales, in [arxiv.org/abs/1106.1144v1](https://arxiv.org/abs/1106.1144v1).
- [10] S. Peng, Falei Wang, BSDE, Path-dependent PDE and Nonlinear Feynman-Kac Formula, in [arxiv.org/abs/1108.4317](https://arxiv.org/abs/1108.4317).